

Cobb Douglas Functions

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1. Introduction

Cobb-Douglas (CD) functions are used extensively in the economics (microeconomic and macroeconomic) literature, mainly because they are simple and tractable and despite the known limitations. The short notes in this document are a refresher in CD functions; those experiencing any difficulties should refer to the standard microeconomic, mathematics for economics and mathematical economics textbooks.

This note provides information on CD utility and production functions that will be used in some of the CGE models in cgemod courses. CGE models also use Constant Elasticity of Substitution/Transformation (CES/CET), nested CES/CET and Stone-Geary (linear expenditure system) functions (LES), and, less commonly, translog functions; CES/CET, LES and nested functions will be introduced as the courses progress.

2. Properties of Cobb-Douglas Production Functions

The CD production function has the general form of

$$X = \alpha \cdot \prod_f F_f^{\beta_f} \quad (2.1)$$

where X is the output, F_f the input/factor of type f , β_f the share parameter for input f and α is the technology parameter. The CD function satisfies all the standard properties of functional forms used in modern economics, and it is straightforward to demonstrate this in the general case. For current purposes, we will operate with a simplified form that only has two inputs - labour (L) and capital (K) – with share parameters β_l and β_k respectively, i.e.,

$$X = \alpha \cdot L^{\beta_l} K^{\beta_k} . \quad (2.2)$$

This simple form is adequate to demonstrate the key properties of the function and can be easily extended to the case of n arguments.

Marginal Products

The change in output resulting from a change in labour input, *ceteris paribus*, is the marginal product of labour, i.e.,

$$\begin{aligned} MP_L &= \frac{\partial X}{\partial L} = \alpha \beta_l L^{\beta_l - 1} K^{\beta_k} \\ &= \beta_l (\alpha L^{\beta_l} K^{\beta_k}) L^{-1} \\ &= \beta_l \frac{X}{L} = \beta_l (AP_L) \end{aligned} \quad (2.3)$$

and is the slope of the production function. It is straightforward to derive the same expression for capital.

The isoquants for a CD function are convex to the origin and satisfy the condition of positive but declining marginal products.

Marginal Rates of (Technical) Substitution

The Marginal Rate of Technical Substitution (MRTS) of labour for capital is the slope of the isoquant, and defines the degree of substitutability, i.e.,

$$\begin{aligned}
 MRTS_{L,K} &= \frac{\partial X / \partial L}{\partial X / \partial K} \\
 &= \frac{\beta_l \cdot X / L}{\beta_k \cdot X / K} \\
 &= \frac{\beta_l}{\beta_k} \cdot \frac{K}{L}
 \end{aligned} \tag{2.4}$$

Elasticity of Substitution

A major weakness of the *MRTS* is its dependence upon the units used to measure inputs. To avoid this an elasticity measure is preferable. The elasticity of substitution, σ (sigma) is defined as the percentage change in the capital labour ratio divided by the percentage change in the *MRTS*. This is a pure number since both numerators and denominators are measured in the same units.

For the CD function the elasticity of substitution is

$$\begin{aligned}
 \sigma &= \frac{\% \Delta \text{ in } K/L}{\% \Delta \text{ in } MRTS} \\
 &= \frac{\partial X / \partial L}{\partial X / \partial K} \cdot \frac{MRTS}{(K/L)} \\
 &= \frac{d(K/L)}{d\left(\frac{\beta_l}{\beta_k} \cdot \left(\frac{K}{L}\right)\right)} \cdot \frac{\beta_l / \beta_k (K/L)}{(K/L)} \\
 &= \frac{\frac{\beta_l}{\beta_k} \cdot d\left(\frac{K}{L}\right)}{\frac{\beta_l}{\beta_k} \cdot d\left(\frac{K}{L}\right)} \\
 &= 1
 \end{aligned} \tag{2.5}$$

This highlights a major concern with the use of the CD functional form: the elasticity of substitution is equal to one, which means there is limited flexibility when using this form. This was a major incentive behind the development of the Constant Elasticity of Substitution (CES) functional form.

Returns to Scale

The distinction between long and short-run analysis is the extent to which factors can be varied. In long-run analysis all factors can be varied. If both factors are increased by the same constant factor k returns to scale identify the relationship between the new level of output (X^*) and the original level of output (X_0).

If

- i) $X^* = kX_0$ - constant returns to scale
- ii) $X^* > kX_0$ - increasing returns to scale
- iii) $X^* < kX_0$ - decreasing returns to scale.

If the production function is homogeneous, the degree of homogeneity, ν , is a measure of returns to scale. If

- i) $\nu = 1$ - constant returns to scale (linear homogeneous)
- ii) $\nu > 1$ - increasing returns to scale
- iii) $\nu < 1$ - decreasing returns to scale.

In the Cobb-Douglas case

$$X_0 = \alpha L^{\beta_l} K^{\beta_k} \quad (2.6)$$

and

$$\begin{aligned} X^* &= \alpha (kL)^{\beta_l} (kK)^{\beta_k} \\ &= (\alpha L^{\beta_l} K^{\beta_k}) k^{(\beta_l + \beta_k)} \\ &= k^{(\beta_l + \beta_k)} X_0 \end{aligned} \quad (2.7)$$

and therefore

$$\nu = \beta_l + \beta_k. \quad (2.8)$$

Thus

$$\begin{aligned} \beta_l + \beta_k < 1 &\Rightarrow DRTS \\ \beta_l + \beta_k &= 1 \Rightarrow CRTS \\ \beta_l + \beta_k > 1 &\Rightarrow IRTS \end{aligned} \quad (2.9)$$

In the vast majority of CGE applications it is assumed that the exponents on CD functions sum to one, i.e., CRTS is assumed. This has certain useful simplifying properties.

Cost Minimisation in the Long Run

For the cost function

$$C = wL + rK \quad (2.10)$$

where w and r are constants, we seek to minimize the cost of producing a given level of output.

This is simply a constrained optimisation procedure, i.e.,

$$\begin{aligned} \text{Min} \quad & C = wL + rK \\ \text{Sto} \quad & \bar{X} = \alpha L^{\beta_l} K^{\beta_k} \end{aligned} \quad (2.11)$$

where $\bar{X}$ indicates that output is fixed, and the production function is CD with CRTS.

Form the Lagrangian

$$\phi = wL + rK + \lambda (X - \alpha L^{\beta_l} K^{\beta_k}) \quad (2.12)$$

and set the derivatives equal to zero

$$\begin{aligned} \frac{\partial \phi}{\partial L} &= w - \lambda \beta_l \alpha L^{\beta_l - 1} K^{\beta_k} = w - \lambda \beta_l \cdot \frac{X}{L} = 0 \\ \frac{\partial \phi}{\partial K} &= r - \lambda \beta_k \alpha L^{\beta_l} K^{\beta_k - 1} = r - \lambda \beta_k \cdot \frac{X}{K} = 0 \quad (2.13) \\ \frac{\partial \phi}{\partial \lambda} &= X - \alpha L^{\beta_l} K^{\beta_k} = 0 \end{aligned}$$

Solving the first two partial derivatives for λ and rearranging gives

$$\frac{w}{\beta_l \cdot \frac{X}{L}} = \lambda = \frac{r}{\beta_k \cdot \frac{X}{K}} \quad (2.14)$$

which can be written as

$$\frac{w}{r} = \frac{\beta_l \cdot \frac{X}{L}}{\beta_k \cdot \frac{X}{K}} = \frac{\beta_l}{\beta_k} \cdot \frac{K}{L} \quad (2.15)$$

which is the standard first order condition (FOC) where the ratio of factor prices equals the *MRTS*. The choice of functional form ensures that the FOCs are optima.

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Starting from (2.15) and solving for K

$$K = \frac{w}{r} \cdot \frac{\beta_k}{\beta_l} \cdot L \quad (2.16)$$

and substituting into the production constraint/function

$$X = \alpha L^{\beta_l} \left(\frac{w}{r} \cdot \frac{\beta_k}{\beta_l} \cdot L \right)^{\beta_k} \quad (2.17)$$

and then solve for L . First rearrange the RHS

$$X = \alpha L^{\beta_l} \left(\frac{w}{r} \cdot \frac{\beta_k}{\beta_l} \cdot L \right)^{\beta_k} = \alpha L^{\beta_l} (L)^{\beta_k} \left(\frac{w}{r} \cdot \frac{\beta_k}{\beta_l} \right)^{\beta_k} = \alpha L \left(\frac{w}{r} \cdot \frac{\beta_k}{\beta_l} \right)^{\beta_k} \quad (2.18)$$

and then solve

$$L = \frac{1}{\alpha \left(\frac{w}{r} \cdot \frac{\beta_k}{\beta_l} \right)^{\beta_k}} \cdot X \quad (2.19)$$

which expresses the cost minimizing quantity of L as a function of X . The same can be done for K , i.e.,

$$K = \frac{1}{\alpha \left(\frac{r}{w} \cdot \frac{\beta_l}{\beta_k} \right)^{\beta_l}} \cdot X \quad (2.20)$$

which expresses the cost minimizing quantity of K as a function of X .

We can now write the long run cost function as

$$\begin{aligned} C(X) &= w \cdot L + r \cdot K \\ &= w \cdot \frac{1}{\alpha \left(\frac{w}{r} \cdot \frac{\beta_k}{\beta_l} \right)^{\beta_k}} + r \cdot \frac{1}{\alpha \left(\frac{r}{w} \cdot \frac{\beta_l}{\beta_k} \right)^{\beta_l}} \cdot X \end{aligned} \quad (2.21)$$

If we give the production function a simple form, such as

$$X = L^{0.5} K^{0.5}$$

then (2.21) simplifies greatly, i.e.,

$$\begin{aligned}
 C(X) &= w \cdot \frac{1}{\left(w \cdot \frac{1}{r} \cdot \frac{0.5}{0.5}\right)^{0.5}} \cdot X + r \cdot \frac{1}{\left(r \cdot \frac{1}{w} \cdot \frac{0.5}{0.5}\right)^{0.5}} \cdot X \\
 &= w \cdot \frac{1}{w^{0.5} \cdot r^{-0.5}} \cdot X + r \cdot \frac{1}{w^{-0.5} \cdot r^{0.5}} \cdot X \\
 &= w \cdot w^{-0.5} \cdot r^{0.5} \cdot X + r \cdot w^{0.5} \cdot r^{-0.5} \cdot X \\
 &= w^{0.5} \cdot r^{0.5} \cdot X + w^{0.5} \cdot r^{0.5} \cdot X = 2(w^{0.5} \cdot r^{0.5} \cdot X)
 \end{aligned} \tag{2.22}$$

Finally, it is useful to see what happens if we differentiate (2.22) with respect to the factor prices. These partial differentials give

$$\begin{aligned}
 \frac{\partial C}{\partial w} &= 2(0.5 w^{-0.5} \cdot r^{0.5} \cdot X) = w^{-0.5} \cdot r^{0.5} \cdot X \\
 \frac{\partial C}{\partial r} &= 2(0.5 w^{0.5} \cdot r^{-0.5} \cdot X) = w^{0.5} \cdot r^{-0.5} \cdot X
 \end{aligned} \tag{2.23}$$

which, since w , r and X must all be positive, are both positive. Hence, as factor prices increase so do costs.

Notice that these derivatives of the cost function with respect to factor prices are identical to the expressions for the cost minimising quantities of the factors, (2.19) and (2.20), when the parameters from the specified production function are substituted. This is not a coincidence: in general, the derivative of the cost function with respect to a factor price is the cost minimizing quantity of that factor.

3. Cobb-Douglas Production Functions in CGE

Consider the activities column in a Social Accounting Matrix. The value of output by each activity is known, and since total income must equal total expenditure the following accounting identity must exist

$$PX_a * QX_a = \sum_f WF_f * FD_{f,a} \quad \forall a \quad (3.1)$$

where WF_f is the price of factor f , $FD_{f,a}$ is the quantity of factor f used by activity a , PX is the basic prices received by activities and QX is the output by activity, and in a perfectly competitive industry there will be zero profit. One way to approach this problem is from the perspective a specific production function and its associated first order conditions for profit maximization. Defining the production function as a Cobb-Douglas function gives

$$QX_a = \alpha_a^x \cdot FD_{l,a}^{\beta_{l,a}} \cdot FD_{k,a}^{\beta_{k,a}} = \alpha_a^x \cdot \prod_f FD_{f,a}^{\beta_{f,a}} \quad (3.2)$$

where α is the efficiency parameter, and if there are constant returns to scale the β coefficients sum to 1. The standard reduced form condition for the profit maximising input ratio with homogenous factors is

$$\frac{\partial QX_a / \partial FD_{l,a}}{\partial QX_a / \partial FD_{k,a}} = \frac{WF_l}{WF_k} = \frac{MP_{l,a}}{MP_{k,a}} \quad (3.3)$$

which can be written as

$$\frac{FD_{k,a}}{FD_{l,a}} = \frac{WF_l}{WF_k} \cdot \left(\frac{\beta_{k,a}}{\beta_{l,a}} \right) = \frac{WF_l}{WF_k} \cdot \left(\frac{1 - \beta_{l,a}}{\beta_{l,a}} \right) \quad (3.4)$$

noting how the choice of a linear homogenous functional form means that only relative input quantities and prices are defined. This leaves the scale of production undetermined (see below).

This does not however solve the problem of determining the optimal factor prices. Defining the profits of each activity as

$$\begin{aligned} \Pi_a &= (PX_a * QX_a) - [(WF_l * FD_{l,a}) + (WF_k * FD_{k,a})] \\ &= (PX_a * (\alpha_a^x \cdot FD_{l,a}^{\beta_{l,a}} \cdot FD_{k,a}^{\beta_{k,a}})) - [(WF_l * FD_{l,a}) + (WF_k * FD_{k,a})] \end{aligned} \quad (3.5)$$

and partially differentiating with respect to the factor quantities gives

$$\begin{aligned}\frac{\partial \Pi_a}{\partial FD_{l,a}} &= \left(PX_a * \left(\alpha_a^x \cdot \beta_{l,a} \cdot FD_{l,a}^{(\beta_{l,a}-1)} \cdot FD_{k,a}^{\beta_{k,a}} \right) \right) - WF_l \\ \frac{\partial \Pi_a}{\partial FD_{k,a}} &= \left(PX_a * \left(\alpha_a^x \cdot \beta_{k,a} \cdot FD_{l,a}^{\beta_{l,a}} \cdot FD_{k,a}^{(\beta_{k,a}-1)} \right) \right) - WF_k\end{aligned}\quad (3.6)$$

Both should equal zero for the zero-profit condition to hold and therefore expressions for the factor prices can be written as

$$\begin{aligned}WF_l &= \left(PX_a * \left(\alpha_a^x \cdot \beta_{l,a} \cdot FD_{l,a}^{(\beta_{l,a}-1)} \cdot FD_{k,a}^{\beta_{k,a}} \right) \right) \\ &= PX_a * \beta_{l,a} * \frac{QX_a}{FD_{l,a}} \\ \Rightarrow WF_l * FD_{l,a} &= \beta_{l,a} \cdot (PX_a * QX_a) \\ WF_k &= \left(PX_a * \left(\alpha_a^x \cdot \beta_{k,a} \cdot FD_{l,a}^{\beta_{l,a}} \cdot FD_{k,a}^{(\beta_{k,a}-1)} \right) \right) \\ &= PX_a * \beta_{k,a} * \frac{QX_a}{FD_{k,a}} \\ \Rightarrow WF_k * FD_{k,a} &= \beta_{k,a} \cdot (PX_a * QX_a)\end{aligned}\quad (3.7)$$

which simply states that the shares of the product of activity a received by labour and capital are equal to the respective coefficients on the factors in the production function. This is a simple application of Eulers theorem.¹

The associated input demand functions are

$$\begin{aligned}FD_{l,a} &= \left(\frac{WF_k}{WF_l} \frac{\beta_{l,a}}{(1-\beta_{l,a})} \right) \cdot FD_{k,a} \\ QX_a &= \alpha_a^x \cdot \left(\left(\frac{WF_k}{WF_l} \frac{\beta_{l,a}}{(1-\beta_{l,a})} \right) \cdot FD_{k,a} \right)^{\beta_{l,a}} \cdot FD_{k,a}^{(1-\beta_{l,a})} \\ FD_{k,a} &= \frac{QX_a}{\alpha_a^x} \cdot \left(\frac{WF_l}{WF_k} \cdot \frac{(1-\beta_{l,a})}{\beta_{l,a}} \right)^{\beta_{l,a}}\end{aligned}\quad (3.8)$$

If there is full employment, then

¹ It is common in some models to find an explicit set of equations that define the zero-profit condition for each activity. These are not needed in this approach since the zero-profit condition is embedded in the derivation of the factor prices.

$$\sum_a FD_{f,a} = FS_f \quad \forall f \quad (3.9)$$

where FS_f is the total supply of factor f . If the total supply of each factor is fixed, then the factor prices are sufficient to determine the allocation of each factor across each activity and the total supply of the factors determines the scale of production.

If either factor quantities or prices are known exogenously then the other can be readily calculated. However, if this information is not available it is possible to adopt a convention, first used by Harberger, whereby the factor prices, WF , are set equal to 1 and hence the (normalised) factor quantities, FD , can be calculated: it is useful to use the term ‘value’ quantities to distinguish between quantities measured this way and ‘physical’ quantities.²

The incomes to the factor accounts, YF_f , can now be defined as

$$YF_f = \sum_a WF_f * FD_{f,a} \quad (3.10)$$

and since total income must equal total expenditure for each account, the total payments from the factor accounts to the household accounts are known.

The Distribution of Factor Incomes

It remains to be determined how these are distributed. Starting from the accounting identities for the factor accounts, it can be seen that

$$\sum_a WF_f * FD_{f,a} = YF_f = \sum_h WF_f * FSH_{h,f} \quad \forall f \quad (3.11)$$

are equal to the payments from the sale of factor services to the factor accounts, where $FSH_{h,f}$ is the supply of factor f by household h , and hence that

$$\begin{aligned} WF_f * \sum_a FD_{f,a} &= WF_f * \sum_h FSH_{h,f} \\ \sum_a FD_{f,a} &= \sum_h FSH_{h,f} \quad \forall f \end{aligned} \quad (3.12)$$

which simply confirms that the total demand for factors by activities, $\sum_a FD_{f,a}$, is equal to the total supply of factors by households, $\sum_h FSH_{h,f}$, under the maintained assumption that the factors are homogenous. It is therefore reasonable to expect that payments to households

² The difficult matter of defining the units used to measure capital will be set aside.

by the factor accounts in respect of the supply of factor services by households are proportionate to the ownership of factor services by households.

The incomes to each household from the sale of factor services, YH_h , are given by

$$YH_h = \sum_f WF_f * FSH_{h,f} \quad \forall h \quad (3.13)$$

which can also be written as

$$YH_h = \sum_f hvash_{h,f} * YF_f \quad \forall h \quad (3.14)$$

where

$$hvash_{h,f} = \frac{FS_{h,f}}{\sum_h FS_{h,f}} = \frac{WF_f * FSH_{h,f}}{\sum_h WF_f * FSH_{h,f}} \quad \forall h, f \quad (3.15)$$

such that under the maintained assumption of factor homogeneity means that the distribution of factor incomes to households is in fixed proportions of factor incomes and that **if** there is an unchanged distribution of factor ownership these proportions are constant.³

Income Distribution

So far, we have avoided any mention of how much of each good, X and Y , goes to each household, A and B . The obvious determinants of these relationships are the levels of income received by A and B , and equally clearly these depend upon the prices paid for the factors and the quantities of the factors owned by each household. But we have already seen that the system only solves for relative prices, and therefore the problem reduces to the quantities of factors owned by each household.

The circular flow requires that total expenditure equals total income in the system, i.e.,

$$p_X.X + p_Y.Y = w.\bar{L} + r.\bar{K} \quad (3.16)$$

and for each consumer total expenditure must also equal total income, i.e.,

$$p_X.X_A + p_Y.Y_A = w.\bar{L}_A + r.\bar{K}_A \quad (3.17a)$$

$$p_X.X_B + p_Y.Y_B = w.\bar{L}_B + r.\bar{K}_B \quad (3.17b)$$

³ Recent versions of the STAGE and ANARRES model families relax this assumption. It is sufficient to note here that most CGE models assume of fixed proportions for the distribution of factor incomes.

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and the assumption of full employment ensures that

$$\bar{L} = L_A + L_B \quad \text{and} \quad \bar{K} = K_A + K_B \quad (3.18)$$

which seems to be 5 equations in four unknowns, where the unknowns are the quantities of labour and capital owned by each household.

But equations (3.17a) and (3.17b) are not independent, by the product exhaustion theorem applied to linear homogenous systems. We can solve this by setting one quantity as a *numéraire* but, unlike the case with prices, this is not a value free presumption – why should an individual's endowment of any factor be fixed, whereas for prices we are dealing with means to ends.

Different distributions of resources will lead to different product combinations and therefore to different general equilibrium solutions. This is critical to the considerations of welfare economics.

4. Cobb Douglas Utility Functions

Cobb-Douglas functions can also be used for utility functions. The utility functions for households, U_h , and hence the commodity demand, QCD , can be written as

$$U_h = \phi_h^u \cdot \prod_c QCD_{c,h}^{\gamma_{c,h}} \quad (4.1)$$

where given the coefficients on the quantities sum to one will ensure that all income is spent and the demand system is complete (the proof is derived using Eulers theorem). The standard reduced form condition for the utility maximising input ratio is

$$\frac{\partial U_h / \partial QCD_{1,h}}{\partial U_h / \partial QCD_{2,h}} = \frac{PQD_1}{PQD_2} = \frac{MU_{1,h}}{MU_{2,h}} \quad (4.2)$$

where PQD is the price of QCD and MU is the marginal utility, and can be written as

$$\frac{QCD_{2,h}}{QCD_{1,h}} = \frac{PQD_1}{PQD_2} \cdot \left(\frac{\gamma_{2,h}}{\gamma_{1,h}} \right) = \frac{PQD_1}{PQD_2} \cdot \left(\frac{1 - \gamma_{1,h}}{\gamma_{1,h}} \right) \quad (4.3)$$

noting how the choice of a linear homogenous functional form means that only relative quantities and prices are defined. This leaves the scale of consumption undetermined (see below).

Considering this as a utility maximisation problem it is straightforward to demonstrate, by an application of Eulers theorem, that provided the exponents sum to one that the expenditure shares are equal to the values of the components. For the utility function in (4.1), the marginal utilities for the 2 commodities for household h are

$$\begin{aligned} \frac{\partial U_h}{\partial QCD_{1,h}} &= \phi_h \cdot \gamma_{1,h} \cdot QCD_{1,h}^{\gamma_{1,h}-1} \cdot QCD_{2,h}^{\gamma_{2,h}} = \gamma_{1,h} \cdot \frac{U_h}{QCD_{1,h}} \\ \frac{\partial U_h}{\partial QCD_{2,h}} &= \phi_h \cdot \gamma_{2,h} \cdot QCD_{1,h}^{\gamma_{1,h}} \cdot QCD_{2,h}^{\gamma_{2,h}-1} = \gamma_{2,h} \cdot \frac{U_h}{QCD_{2,h}} \end{aligned} \quad (4.4)$$

And from the first order conditions

$$\frac{PQD_1}{PQD_2} = \gamma_{1,h} \cdot \frac{U_h}{QCD_{1,h}} \bigg/ \gamma_{2,h} \cdot \frac{U_h}{QCD_{2,h}} = \frac{MU_{QCD_{1,h}}}{MU_{QCD_{2,h}}} \quad (4.5)$$

or,

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$$\frac{PQD_1}{PQD_2} = \frac{\gamma_{1,h}}{\gamma_{2,h}} \cdot \frac{QCD_{2,h}}{QCD_{1,h}} \quad (4.6)$$

The household budget constraint, where YH is household income, is

$$YH_h = PQD_1 \cdot QCD_{1,h} + PQD_2 \cdot QCD_{2,h} \quad (4.7)$$

Optimal consumption, combined with the budget constraint, yields the demand equations:

$$YH_h = PQD_1 \cdot QCD_{1,h} + PQD_2 \cdot \left(\frac{PQD_1}{PQD_2} \cdot \frac{\gamma_{2,h}}{\gamma_{1,h}} \cdot QCD_{1,h} \right) \quad (4.8)$$

$$QCD_{1,h} = \gamma_{1,h} \cdot \frac{YH_h}{PQD_1}$$

i.e., the value of commodity 1 consumed by household h is equal to the share of expenditure on commodity 1 by household h .

For the general case of c commodities where $\gamma_{c,h}$ is the expenditure share on commodity c by household h , we can define

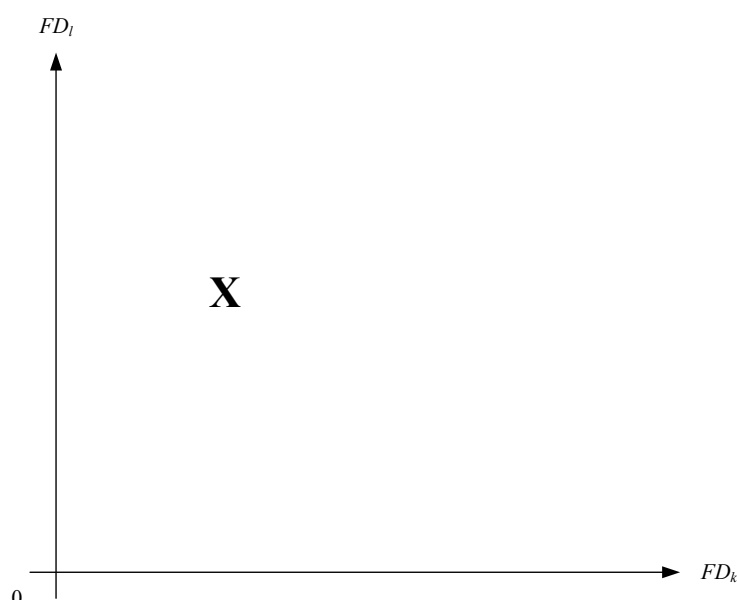
$$\sum_c PQD_c \cdot QCD_{c,h} = \sum_c \gamma_{c,h} \cdot YH_h = YH_h \quad (4.9)$$

which means that if YH and the expenditure shares are known then QCD is defined. This is sufficient to ensure that all income, or as will prove useful all disposable income, is spent on consumption. Thus, once household incomes/disposable incomes/consumption expenditures are determined so is the scale of consumption.

5. Calibrating Cobb Douglas Functions in CGE Models

The problem of calibration in CGE models is that the observed data only identify transactions values, whereas the models need to identify, separately, the prices and quantities of inputs and outputs and the parameters of the chosen functional forms. For the case of two inputs and one output the problem can be summarised with a simple diagram (Figure 1), where **X** marks the observed transaction values of the two inputs. The problem thus reduces to identifying the prices of the two inputs, the price and quantity of the output and the parameters of the production function.

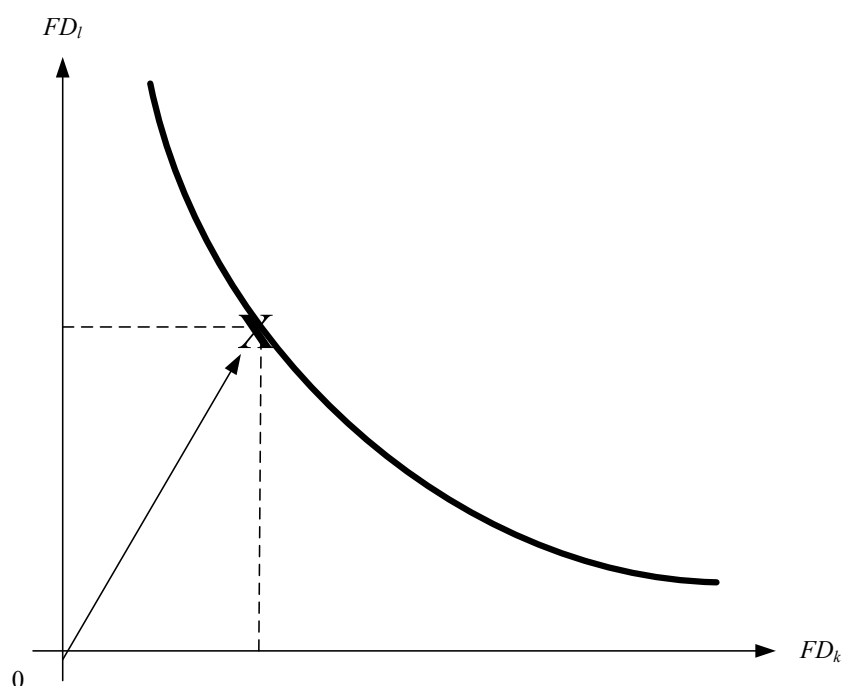
Figure 1 Calibration Problem for CGE Models



Since the functional forms used are all linear homogeneous the model is defined in terms of relative prices. Thus, if as is often the case, we do not have data for prices, we can assign arbitrary prices for the inputs and outputs. Typically, the prices will be set equal to one so the quantity units are ‘value’ quantities. Thus, the problem is further reduced to the identification of the parameters of the production function: namely the elasticity of substitution, the shift or efficiency parameter and the share parameters.

Given the transactions data for the two inputs, the elasticity of substitution and the assumption of technical and economic efficiency, the point X in the diagram must lie on the isoquant that represents the level of the output and the curvature of the isoquant is known.⁴

Figure 2 Calibration in CGE Models



The calibration process works backwards. The modeler must tell the programme the type of functional form, e.g., CD, CES, etc., the transactions values, the prices and the elasticity of substitution. Given this information, and the appropriate equations, the model can determine the quantities of output/input, and the share and shift parameters for the function.

In the case of the CD function the specifying the functional form also specifies the elasticity of substitution, i.e., 1 (see 2.5 above). The share parameters for the CD function can then be defined; if the Cobb-Douglas function has constant returns to scale then the product exhaustion theorem holds so the share parameters are the coefficients on the arguments (see 4.7 above). That leaves the shift parameter, which tells the model ‘how far’ the isoquant/indifference curve is from the origin. Given the transaction value of the output and the price of the output then the level of output (X) can be defined. Given this and the previous

⁴ Strictly, it is not necessary to assume that the system operates of the efficient boundary, but is necessary to assume that proportionate divergence from the boundary is constant; i.e., the relative degree of (in)efficiency is constant.

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calculations the CD function can be rearranged with the shift parameter on the LHS and solved. In GAMS the code for this looks like this

$$\alpha(f, a) = \text{SAM}(f, a) / \text{SUM}(fp, \text{SAM}(fp, a)) ;$$

$$ad(a) = QX0(a) / \text{PROD}(f, FD0(f, a) ** \alpha(f, a)) ;$$

where *alpha* is the shift parameter and *ad* the shift parameter, *QX0* the base level output, *FD0* the base level factor demand, and *PROD* indicates a product.