The SAM-Leontief Model

Contents

1. Introduction

The SAM-Leontief model is a descendant of the Leontief input-output model. The model has spawned a large, and ongoing, literature that encompasses a wide range of comparative static and dynamic multiplier models and associated decompositions. Members of this family of models assume, in some fashion or other, fixed prices or fixed relative quantities, and this is among the reasons that CGE have tended to supersede Leontief models. In some respects, this may be justified while in others it may not be justified. However, whether this is justified, it cannot be denied that the Leontief models still have some benefits: they are simple and quick to implement and can provide useful policy analyses if their limitations are recognized; they are powerful descriptive tools for the analyses of economic structures; they can provide valuable information about the price formation processes in all whole economy models; and they provide a pedagogic tool for learning how to move from a data system, a SAM, to an economic model.

The objectives in this paper is to provide introductions to three variants of the SAM-Leontief model; SAM income multipliers, SAM price multipliers and SAM mixed multipliers. The section, two, on income multipliers introduces the tools used in multiplier models and demonstrates how the patterns of income distribution influence the results. Price multipliers, covered in section three, starts from the price dual and demonstrates how the column coefficients in a SAM-Leontief model drive the price formation processes. Both the income and price multiplier models are limited by the assumption of, respectively, fixed price and quantity weights. The final section considers a variant of the income multiplier model that relaxes the assumption that the supplies of all factors are infinitely elastic; this is the so-called mixed multiplier model.

2. SAM Income Multipliers

The move from using a SAM as system for the organisation of data to a SAM based model requires certain decisions to be made. The simplest way to see this is to set up a simple SAM-Leontief (linear) economic model. This involves assuming all the behavioural relationships can be specified as linear functions and that the system is characterised by excess capacity (this aspect of the approach is classically Keynesian). These types of model can be used to examine ‘linkages’ within an economic system, and have great advantages as descriptive tools. Having decided upon the functional relations of the model, it is necessary to determine which of the accounts are to be exogenous and which are to be endogenous. In the standard Leontief input-output model this is straightforward: the model is a demand driven model and hence the final demand account (**d**) is exogenous, technology is given by the technical coefficients and hence gross outputs (**q**) are determined endogenously.

The core of the Leontief model is the materials balance equation

$$\begin{aligned} \mathbf{q} &= \mathbf{Aq} + \mathbf{d} \\ \mathbf{q} - \mathbf{Aq} &= \mathbf{d} \\ \mathbf{q} &= \left[\mathbf{I} - \mathbf{A}\right]^{-1} \mathbf{d} \end{aligned} \tag{1}$$

which defines gross outputs (**q**) as the sum of endogenously determined output (**Aq**) plus exogenously determined output (**d**)

We have more choice with a SAM model. Typically, the choice is to use one or more of the capital, government or rest of the world accounts where the choice is based on macroeconomic theory (closure rule) and/or the issue being examined.

In the endogenous accounts, matrix $\mathbf{N}_{21}$ distributes value added by productive activities to factors of production; $\mathbf{N}_{11}$ is the production matrix; $\mathbf{N}_{32}$ captures the socio-economic characteristics in the form of income distribution; $\mathbf{N}_{33}$ maps income transfers amongst households; and $\mathbf{N}_{13}$ shows the expenditure patterns by households. All these matrices can be said to be subsets of a partitioned endogenous expenditure matrix, **N**.

A ‘shock’ to the system can then be introduced through the column of exogenous accounts and the ‘leakages’ from the system are recorded through the rows of the exogenous accounts. This is equivalent to the injections/investment equal to withdrawals/savings equilibrium condition in the Keynesian income and expenditure model. The model is solved by the determination of the equilibrium levels for all the endogenous accounts. If a single exogenous account is chosen it will by necessity balance in equilibrium. If more than one account is deemed exogeneous then it is the aggregate of the exogenous accounts, which must balance. The multipliers computed will depend upon the choice of exogenous and endogenous accounts. By suitably choosing the designation of exogenous accounts several policy options and/or planning scenarios can be considered. For instance, the following could be considered:

i) Rest of the world - trade, aid, foreign remittances etc.

ii) Capital account - investment

iii) Government - tax policy, transfers policy etc.

The literature on SAM multipliers has had two major focuses. First to revise the view of development propounded by the early linkages literature in light the information provided by ‘models’, which represent the full circular flow of income. And second, decompositions of SAM multipliers that distinguish between the effects of the various components of the circular flow.

The core of the SAM

Table 1.1 Simplified Schematic Social Accounting Matrix

			Expenditures				
			Endogenous accounts			Exogenous accounts	
			1	2	3	4	5
			Commodities Activities	Factors	Households	Government Investment Rest of World	Totals
Incomes	Endogenous accounts	1	Commodities Activities	$\mathbf{N}_{11}$	$\mathbf{0}$	$\mathbf{N}_{13}$	$\mathbf{x}_1$	$\mathbf{y}_1$
		2	Factors	$\mathbf{N}_{21}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{x}_2$	$\mathbf{y}_2$
		3	Households	$\mathbf{0}$	$\mathbf{N}_{32}$	$\mathbf{N}_{33}$	$\mathbf{x}_3$	$\mathbf{y}_3$
	Exogenous accounts	4	Government Investment Rest of World	$\mathbf{l}_1'$	$\mathbf{l}_2'$	$\mathbf{l}_3'$	$\mathbf{t}$	$\mathbf{y}_4$
		5	Totals	$\mathbf{y}_1'$	$\mathbf{y}_2'$	$\mathbf{y}_3'$	$\mathbf{y}_4'$	

<u>Accounting Multipliers</u>

Table 1.2 sets out a SAM and summarises the notation.

Table 1.2 Notation and Accounting Balances

	Expenditures		
Incomes	**Endogenous Accounts**	**Exogenous Accounts**	**Totals**
Endogenous Accounts	$\mathbf{N} = \mathbf{A}_n \hat{\mathbf{y}}_n$ (2)	$\mathbf{X}$	$\mathbf{y}_n = \mathbf{n} + \mathbf{x}$ $= \mathbf{A}_n \mathbf{y}_n + \mathbf{x}$ (4)
Exogenous Accounts	$\mathbf{L} = \mathbf{A}_l \hat{\mathbf{y}}_n$ (3)	$\mathbf{R}$	$\mathbf{y}_x = \mathbf{l} + \mathbf{R}\mathbf{i}$ $= \mathbf{A}_l \mathbf{y}_n + \mathbf{R}\mathbf{i}$ (5)
Totals	$\mathbf{y}'_n = (\mathbf{i}'\mathbf{A}_n + \mathbf{i}'\mathbf{A}_l)\hat{\mathbf{y}}_n$ (6)	$\mathbf{y}'_x = \mathbf{i}'\mathbf{X} + \mathbf{i}'\mathbf{R}$ (8)	
	$\therefore \mathbf{i}' = \mathbf{i}'\mathbf{A}_n + \mathbf{i}'\mathbf{A}_l$ (7)	$\therefore \mathbf{A}_l \mathbf{y}_n - \mathbf{X}'\mathbf{i} = (\mathbf{R} - \mathbf{R}')\mathbf{i}$ (9)	$\lambda'_a \mathbf{y}_n = \mathbf{x}'\mathbf{i}$ (10)

$\mathbf{A}_n = \mathbf{N}\hat{\mathbf{y}}_n^{-1}$ = matrix of average endogenous expenditure propensities;
$\mathbf{A}_l = \mathbf{L}\hat{\mathbf{y}}_n^{-1}$ = matrix of average propensities to leak;
$\mathbf{Ni} = \mathbf{n}$ = vector of row sums of $\mathbf{N} = \mathbf{A}_n \hat{\mathbf{y}}_n \rightarrow \mathbf{n} = \mathbf{A}_n \mathbf{y}_n$;
$\mathbf{Xi} = \mathbf{x}$ = vector of row sums of $\mathbf{X}$;
$\mathbf{Li} = \mathbf{l}$ = vector of row sums of $\mathbf{L} = \mathbf{A}_l \hat{\mathbf{y}}_n \rightarrow \mathbf{l} = \mathbf{A}_l \mathbf{y}_n$;
$\lambda'_a = \mathbf{i}'\mathbf{A}_l$ = vector of column sums of $\mathbf{A}_l$, i.e., the vector of aggregate average propensities to leak;
$\mathbf{N}$ = the matrix of SAM transactions between endogenous accounts;
$\mathbf{X}$ = the matrix of injections from exogenous into endogenous accounts;
$\mathbf{L}$ = the matrix of leakages from endogenous into exogenous accounts;
$\mathbf{R}$ = the matrix of SAM transactions between exogenous accounts.

From the Leontief materials balance relationship (1) and the notation in Table 2.1 equation 4 we get a matrix of SAM multipliers

$$\begin{aligned} \mathbf{y}_n &= \mathbf{A}_n\mathbf{y}_n + \mathbf{x} \\ \mathbf{y}_n - \mathbf{A}_n\mathbf{y}_n &= \mathbf{x} \\ (\mathbf{I} - \mathbf{A}_n)\mathbf{y}_n &= \mathbf{x} \\ \mathbf{y}_n &= (\mathbf{I} - \mathbf{A}_n)^{-1}\mathbf{x} \\ \mathbf{y}_n &= \mathbf{M}_a\mathbf{x} \end{aligned} \tag{11}$$

which can also be applied to leakages

$$\begin{aligned} \mathbf{l} &= \mathbf{A}_l\mathbf{y}_n \\ &= \mathbf{A}_l(\mathbf{I} - \mathbf{A}_n)^{-1}\mathbf{x} \\ &= \mathbf{A}_l\mathbf{M}_a\mathbf{x} \end{aligned} \tag{12}$$

and the archetypal equilibrium condition of Keynesian macroeconomics, ‘injections equal to withdrawals’, is given by

$$\left[(\mathbf{i}'\mathbf{A}_n + \mathbf{i}'\mathbf{A}_l)\hat{\mathbf{y}}_n\right]'\mathbf{i} = \mathbf{y}_n'\mathbf{i} = \mathbf{i}'(\mathbf{A}_n\mathbf{y}_n + \mathbf{x}) = \mathbf{i}'\mathbf{y}_n . \tag{13}$$

Interpreting the Multiplier Matrix

An understanding of the material balance equation of the SAM-Leontief model requires an understanding of the multiplier matrix. The (Leontief) inverse matrix can be expressed as a converging series expansion, i.e.,

$$\mathbf{M}_a = (\mathbf{I} - \mathbf{A}_n)^{-1} = \mathbf{I} + \mathbf{A}_n + \mathbf{A}_n^2 + \mathbf{A}_n^3 + \mathbf{A}_n^4 + ... \tag{14}$$

and as the exponent increases so the approximation becomes closer,[1] and the material balance equation can be written as

$$\mathbf{y} \approx (\mathbf{I} + \mathbf{A}_n + \mathbf{A}_n^2 + \mathbf{A}_n^3 + \mathbf{A}_n^4 + ...)\mathbf{x} = \mathbf{x} + \mathbf{A}_n\mathbf{x} + \mathbf{A}_n^2\mathbf{x} + \mathbf{A}_n^3\mathbf{x} + \mathbf{A}_n^4\mathbf{x} + ... \tag{15}$$

The series expansion approximation to the materials balance equation proves useful to understanding the fundamental nature of the operation of multisector economic models. A SAM-Leonteif model has a very Keynesian flavour: the vector **x** is presumed to be exogenously determined and the material balance equation allows the endogenous determination of the vector **y** that is consistent with the new equilibrium. As such it can be

[1] For many practical purposes i = 5 is often sufficiently accurate.

used to facilitate planning exercises which are concerned with the practicality of various exogenously determined targets.

To better appreciate the 'ripple' effects of interdependency it is convenient to examine the implications for the endogenous accounts of producing additional units of the exogenous vector. The required increase in economic activity is determined as a series of rounds of productive activity, which can be written as

$$\Delta\mathbf{y} \approx \Delta\mathbf{x} + \mathbf{A}_n\Delta\mathbf{x} + \mathbf{A}_n^2\Delta\mathbf{x} + \mathbf{A}_n^3\Delta\mathbf{x} + \mathbf{A}_n^4\Delta\mathbf{x} + ... \quad (16)$$

where Δ indicates change. The first round is the required increase in **x**, i.e., $\Delta\mathbf{x}$, which requires additional economic activity for its production, $\mathbf{A}_n\Delta\mathbf{x}$, but this extra economic activity also requires additional economic activity for its production, i.e., $\mathbf{A}_n^2\Delta\mathbf{x}$, and so on. All entries in the matrix A are less that one and all the column totals are less than one, thus as the exponent on **A** increases so the magnitude of the product decreases, i.e.,

$$\mathbf{A}_n > \mathbf{A}_n^2 > \mathbf{A}_n^3 > \mathbf{A}_n^4 > ....$$

An immediate consequence of such interdependency is the fact that changes in demand for the characteristic product of an industry have implications for the supply of products by a whole range, if not all, of the industries in an economy. This feeds through into the demand for factors by activities and thence into incomes by households and so on through the system. Thus, for example, policy changes directed at specific industries, which encourage or discourage their activities, and/or shocks that affect exports of a commodity, will have implications for the clear majority of industries in an economy. These implications are typically ignored in macroeconomic and partial equilibrium models. Hence, while the assumptions of linear behavioural functions and lack of substitution possibilities may limit a SAM-Leontief model, it does recognise the effects of interdependency.

Income Distribution, Consumption Patterns and Production

It is useful to explore further the material balance relationship by means of matrix partitioning; this gives insights into a fundamental difference between the SAM-Leontief and IO-Leontief models. Based on the information in Tables 2.1 & 2.2, equation (4) can be rewritten as

$$\begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{y}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} & \mathbf{A}_{13} \\ \mathbf{A}_{21} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{32} & \mathbf{A}_{33} \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{y}_3 \end{bmatrix} + \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix} \tag{17}$$

which can then be solved as a set of simultaneous equations

$$\begin{aligned} \mathbf{y}_1 &= \mathbf{A}_{11}\mathbf{y}_1 + \mathbf{A}_{13}\mathbf{y}_3 + \mathbf{x}_1 \\ \mathbf{y}_2 &= \mathbf{A}_{21}\mathbf{y}_1 + \mathbf{x}_2 \\ \mathbf{y}_3 &= \mathbf{A}_{32}\mathbf{y}_2 + \mathbf{A}_{33}\mathbf{y}_3 + \mathbf{x}_3 \end{aligned} . \tag{18}$$

The first line of (18) refers to the production accounts, i.e., $\mathbf{y}_1$ is the output of production activities, which is the vector of total outputs from the corresponding IO table. We can therefore solve the first line of (18) for $\mathbf{y}_1$

$$\begin{aligned} \mathbf{y}_1 &= \mathbf{A}_{11}\mathbf{y}_1 + \mathbf{A}_{13}\mathbf{y}_3 + \mathbf{x}_1 \\ \mathbf{y}_1 - \mathbf{A}_{11}\mathbf{y}_1 &= \mathbf{A}_{13}\mathbf{y}_3 + \mathbf{x}_1 \\ \mathbf{y}_1 &= \left(\mathbf{I} - \mathbf{A}_{11}\right)\left(\mathbf{A}_{13}\mathbf{y}_3 + \mathbf{x}_1\right) \end{aligned} \tag{19}$$

and compare this representation of the IO materials balance with the standard representation, e.g.,

$$\mathbf{q} = \left[\mathbf{I} - \mathbf{A}\right]^{-1}\mathbf{d}$$

and therefore

$$\begin{aligned} \mathbf{q} &= \mathbf{y}_1 \\ \left(\mathbf{I} - \mathbf{A}\right)^{-1} &= \left(\mathbf{I} - \mathbf{A}_{13}\right)^{-1} \\ \mathbf{d} &= \left(\mathbf{A}_{13}\mathbf{y}_3 + \mathbf{x}_1\right) \end{aligned} . \tag{20}$$

This demonstrates that the difference between the IO and the SAM representation of the IO material balance relates solely to the treatment of final demands (**d**), and more particularly, for this specification of the exogenous accounts, to the treatment of final demands by households. Final demand by households is a product of household income levels ($\mathbf{y}_2$) and the patterns of consumption expenditure by households' ($\mathbf{A}_{13}$). If the distribution of household income changes, i.e., the structure of the vector $\mathbf{y}_2$ changes, then even if the total of household income remains constant, i.e., $\mathbf{i}'\mathbf{y}_2 = c$, the pattern of final demands will change. However, if the patterns of consumption for all types of household are the identical, i.e., the structures of the columns of $\mathbf{A}_{13}$ are the same, the pattern of final demands by those households will be unaffected by redistribution. If that was the case, it may be asked why separate types of household were identified, since separate accounts for different households indicate that the households are homogenous. The reasons for identifying different household groups is the observation that different groups of households' have different patterns of expenditure ($\mathbf{A}_{13}$) and different patterns of income.

Consequently, the levels of output by the production activities in an economic system, the vector $\mathbf{y}_3$, are influenced by the distribution of income.

This is an insight into an important dimension of multi-sector economic models. In all SAM based economic models with multiple household/institution accounts, the distribution of income is relevant to the results generated by the models. There is one important proviso to this. If the factors accounts are not disaggregated in an 'appropriate' manner, then the information about income distribution generated by such models will be limited.

Fixed Price Multipliers

Accounting multipliers presume that the average coefficients are the same as the marginal coefficients. If nothing else this conflicts with Engels Law.

Starting from the accounting balance equation

$$\mathbf{y}_n = \mathbf{n} + \mathbf{x} \tag{21}$$

it follows that

$$\begin{aligned} d\mathbf{y}_n &= d\mathbf{n} + d\mathbf{x} \\ &= \mathbf{C}_n d\mathbf{y}_n + d\mathbf{x} \end{aligned} \tag{22}$$

where $\mathbf{C}_n$ is a matrix of marginal coefficients. Solving for $d\mathbf{y}_n$

$$\begin{aligned} d\mathbf{y}_n &= \mathbf{C}_n d\mathbf{y}_n + d\mathbf{x} \\ d\mathbf{y}_n - \mathbf{C}_n d\mathbf{y}_n &= d\mathbf{x} \\ (\mathbf{I} - \mathbf{C}_n) d\mathbf{y}_n &= d\mathbf{x} \\ d\mathbf{y}_n &= (\mathbf{I} - \mathbf{C}_n)^{-1} d\mathbf{x} \\ d\mathbf{y}_n &= \mathbf{M}_c d\mathbf{x} \end{aligned} . \tag{23}$$

and the interpretation remains as before but now relates to marginal changes rather than average changes.

Relationship between Average and Marginal Coefficients

The relationships between the average and marginal coefficients are determined by the income elasticities of demand, i.e.,

$$a_{ij} = \frac{q_{ij}}{Q_j} \tag{24}$$

and the income elasticity is defined as

$$\begin{aligned} \eta_{ij} &= \frac{dq_{ij}/q_{ij}}{dY_j/Y_j} \\ &= \frac{dq_{ij}}{dY_j}.\frac{Y_j}{q_{ij}} \\ &= \frac{dq_{ij}}{dY_j}.\frac{1}{a_{ij}} \end{aligned} \tag{25}$$

since the column sum (Q_j) represents the total income (Y_j) to that account.

It would seem reasonable to assume that the income elasticities of demand might vary across different household groups. Thus, the differences between the $\mathbf{A}_n$ and $\mathbf{C}_n$ matrix might reasonably be expected to differ for the submatrix A13, i.e., the patterns of household demands/preferences, as asserted by Engels Law.

3. SAM Price Multipliers

The archetypal SAM model starts from the material balance relationship and views the system from the perspective of the primal, or quantities, perspective. This entails a perception of the system as a process whereby incomes are generated consequent upon the system being divided into series of endogenous and exogenous accounts: as such the model is a simple extension of the standard input-output model. However, the model can be reformulated from the perspective of the price dual, and then used to examine the price formation and cost transmission mechanisms (see Roland-Holst and Sancho, 1995). Clearly this requires the imposition of restrictive conditions similar to those required for conventional SAM models, provided those are reasonably acceptable, the resultant models provide useful complementary analyses.

Table 1 Simplified Schematic Social Accounting Matrixs

			Expenditures: Endogenous accounts 1 Commodities, Activities	Expenditures: Endogenous accounts 2 Factors	Expenditures: Endogenous accounts 3 Households	Expenditures: Exogenous accounts 4 Government, Investment, Rest of World	5 Totals
Incomes: Endogenous accounts	1	Commodities, Activities	$\mathbf{N}_{11}$	$\mathbf{0}$	$\mathbf{N}_{13}$	$\mathbf{x}_1$	$\mathbf{y}_1$
Incomes: Endogenous accounts	2	Factors	$\mathbf{N}_{21}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{x}_2$	$\mathbf{y}_2$
Incomes: Endogenous accounts	3	Households	$\mathbf{0}$	$\mathbf{N}_{32}$	$\mathbf{N}_{33}$	$\mathbf{x}_3$	$\mathbf{y}_3$
Incomes: Exogenous accounts	4	Government, Investment, Rest of World	$\mathbf{l}_1'$	$\mathbf{l}_2'$	$\mathbf{l}_3'$	$\mathbf{t}$	$\mathbf{y}_4$
Incomes	5	Totals	$\mathbf{y}_1'$	$\mathbf{y}_2'$	$\mathbf{y}_3'$	$\mathbf{y}_4'$	

<u>Price Dual in a SAM</u>

The simplified schematic SAM in Table 1 (reproduced above) can be re-written in terms which make explicit the fact that the recorded transactions encompass both quantities and prices, i.e.,

	1	2	3	4	5
Production	$\hat{\mathbf{p}}_1'\mathbf{Q}_{11}$	$\mathbf{0}$	$\hat{\mathbf{p}}_1\mathbf{Q}_{13}$	$\hat{\mathbf{p}}_1'\mathbf{Q}_{14}$	$\hat{\mathbf{p}}_1'\mathbf{q}_1$
Factors	$\hat{\mathbf{p}}_2'\mathbf{Q}_{21}$	$\mathbf{0}$	$\mathbf{0}$	$\hat{\mathbf{p}}_2'\mathbf{Q}_{24}$	$\hat{\mathbf{p}}_2'\mathbf{q}_2$
Households	$\mathbf{0}$	$\hat{\mathbf{p}}_3'\mathbf{Q}_{32}$	$\hat{\mathbf{p}}_3'\mathbf{Q}_{33}$	$\hat{\mathbf{p}}_3'\mathbf{Q}_{34}$	$\hat{\mathbf{p}}_3'\mathbf{q}_3$
Exogenous	$\hat{\mathbf{p}}_4'\mathbf{Q}_{41}$	$\hat{\mathbf{p}}_4'\mathbf{Q}_{42}$	$\hat{\mathbf{p}}_4'\mathbf{Q}_{43}$	$\hat{\mathbf{p}}_4'\mathbf{Q}_{44}$	$\hat{\mathbf{p}}_4'\mathbf{q}_4$
Totals	$\mathbf{p}_1'\hat{\mathbf{q}}_1$	$\mathbf{p}_2'\hat{\mathbf{q}}_2$	$\mathbf{p}_3'\hat{\mathbf{q}}_3$	$\mathbf{p}_4'\hat{\mathbf{q}}_4$	

Defining the technical coefficients as

$$a_{ij} = \frac{Q_{ij}}{q_j} \qquad \textbf{or} \qquad Q_{ij} = a_{ij} q_j \tag{1}$$

the transactions matrix can be rewritten as

	1	2	3	4	5
Production	$\hat{\mathbf{p}}_1'\mathbf{A}_{11}\hat{\mathbf{q}}_1$	$\mathbf{0}$	$\hat{\mathbf{p}}_1\mathbf{A}_{13}\hat{\mathbf{q}}_3$	$\hat{\mathbf{p}}_1'\mathbf{A}_{14}\hat{\mathbf{q}}_4$	$\hat{\mathbf{p}}_1'\mathbf{q}_1$
Factors	$\hat{\mathbf{p}}_2'\mathbf{A}_{21}\hat{\mathbf{q}}_1$	$\mathbf{0}$	$\mathbf{0}$	$\hat{\mathbf{p}}_2'\mathbf{A}_{24}\hat{\mathbf{q}}_4$	$\hat{\mathbf{p}}_2'\mathbf{q}_2$
Households	$\mathbf{0}$	$\hat{\mathbf{p}}_3'\mathbf{A}_{32}\hat{\mathbf{q}}_2$	$\hat{\mathbf{p}}_3'\mathbf{A}_{33}\hat{\mathbf{q}}_3$	$\hat{\mathbf{p}}_3'\mathbf{A}_{34}\hat{\mathbf{q}}_4$	$\hat{\mathbf{p}}_3'\mathbf{q}_3$
Exogenous	$\hat{\mathbf{p}}_4'\mathbf{A}_{41}\hat{\mathbf{q}}_1$	$\hat{\mathbf{p}}_4'\mathbf{A}_{42}\hat{\mathbf{q}}_2$	$\hat{\mathbf{p}}_4'\mathbf{A}_{43}\hat{\mathbf{q}}_3$	$\hat{\mathbf{p}}_4'\mathbf{A}_{44}\hat{\mathbf{q}}_4$	$\hat{\mathbf{p}}_4'\mathbf{q}_4$
Totals	$\mathbf{p}_1'\hat{\mathbf{q}}_1$	$\mathbf{p}_2'\hat{\mathbf{q}}_2$	$\mathbf{p}_3'\hat{\mathbf{q}}_3$	$\mathbf{p}_4'\hat{\mathbf{q}}_4$	

and dividing through by $\mathbf{q}_i$ as appropriate, i.e., by the columns, gives

	1	2	3	4	5
Production	$\hat{\mathbf{p}}_1'\mathbf{A}_{11}$	$\mathbf{0}$	$\hat{\mathbf{p}}_1\mathbf{A}_{13}$	$\hat{\mathbf{p}}_1'\mathbf{A}_{14}$	$\hat{\mathbf{p}}_1$
Factors	$\hat{\mathbf{p}}_2'\mathbf{A}_{21}$	$\mathbf{0}$	$\mathbf{0}$	$\hat{\mathbf{p}}_2'\mathbf{A}_{24}$	$\hat{\mathbf{p}}_2$
Households	$\mathbf{0}$	$\hat{\mathbf{p}}_3'\mathbf{A}_{32}$	$\hat{\mathbf{p}}_3'\mathbf{A}_{33}$	$\hat{\mathbf{p}}_3'\mathbf{A}_{34}$	$\hat{\mathbf{p}}_3$
Exogenous	$\hat{\mathbf{p}}_4'\mathbf{A}_{41}$	$\hat{\mathbf{p}}_4'\mathbf{A}_{42}$	$\hat{\mathbf{p}}_4'\mathbf{A}_{43}$	$\hat{\mathbf{p}}_4'\mathbf{A}_{44}$	$\hat{\mathbf{p}}_4$
Totals	$\mathbf{p}_1'$	$\mathbf{p}_2'$	$\mathbf{p}_3'$	$\mathbf{p}_4'$	

the resultant column identities are then

$$\begin{aligned}\mathbf{p}_1 &= \mathbf{p}_1'\mathbf{A}_{21} + \mathbf{p}_2'\mathbf{A}_{21} + \mathbf{p}_4'\mathbf{A}_{41} \\ \mathbf{p}_2 &= \mathbf{p}_3'\mathbf{A}_{32} + \mathbf{p}_4'\mathbf{A}_{42} \\ \mathbf{p}_3 &= \mathbf{p}_1'\mathbf{A}_{13} + \mathbf{p}_3'\mathbf{A}_{33} + \mathbf{p}_4'\mathbf{A}_{43} \\ \mathbf{p}_4 &= \mathbf{p}_1'\mathbf{A}_{14} + \mathbf{p}_2'\mathbf{A}_{24} + \mathbf{p}_3'\mathbf{A}_{34} + \mathbf{p}_4'\mathbf{A}_{44}\end{aligned} \tag{2}$$

Letting the matrices $\mathbf{A}_{4i}$ for $i = 1, 2, 3$, be row vectors and $\mathbf{p}_4$ be a scalar, i.e., a 'weighted' average price, the vector of exogenous costs, **v**, is

$$\mathbf{v} = p_4\mathbf{a}_4 \tag{3}$$

where $\mathbf{a}_4$ is formed from the row adjoining of the matrices $\mathbf{A}_{4i}$, i.e., $\mathbf{a}_4 = \mathbf{i}'\left[\mathbf{A}_{41}, \mathbf{A}_{42}, \mathbf{A}_{43}, \mathbf{A}_{44}\right]$.

Further defining

$$\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) \tag{4}$$

the price dual can be written as

$$\begin{aligned}\mathbf{p}' &= \mathbf{p}'\mathbf{A} + \mathbf{v}' \\ &= \mathbf{v}'\left[\mathbf{I} - \mathbf{A}_n\right]^{-1} \\ &= \mathbf{v}'\mathbf{M}_p \\ &= \mathbf{M}_p{}'\mathbf{v}\end{aligned} \tag{5}$$

where

$$\mathbf{A} = \begin{bmatrix}\mathbf{A}_{11} & \mathbf{0} & \mathbf{A}_{13} \\ \mathbf{A}_{21} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{32} & \mathbf{A}_{33}\end{bmatrix} \tag{6}$$

This demonstrates that there is an alternative interpretation of the multiplier matrix, which can be achieved by reading across the rows of $\mathbf{M}_p$, rather than down the columns. An exogenous demand driven injection into a sector generates a series of income changes across sectors, which are identified by the relevant column elements of the multiplier matrix. On the other hand, an exogenous increase in the price or cost faced by a sector is transmitted through the economy and the effects are identified by the row elements of the multiplier matrix.

This is a very important insight for all whole economy models. The prices are defined by reference to the column coefficients, cost shares, of the SAM, i.e., the column coefficients are important to the operation of any whole economy model.

4. Mixed Multiplier Model

The standard demand driven SAM-Leontief model presumes the existence of excess capacity in an economic system. Consequently, the predicted response of the economy to an increase in exogenously determined final demands is assessed under the assumption of an absence of supply constraints. However there are circumstances in which it may be important to allow for supply constraints, e.g., the ability of a system to respond to increases in final demand for agricultural products may be constrained by availability of land. This type of situation can be addressed using a model with mixed endogenous/exogenous varibales (see Miller and Blair, 1985; Subramanian and Sadoulet, 1990; Lewis and Thorbecke, 1992; Rich *et al*., 1997).

Mixed Multipliers in a SAM-Leontief Model

SAM-Leontief models are all based on variants of the materials balance relationship (see Pyatt and Round, 1979); all behavioural relationships are linear and the models are 'open with respect to final demand'. It is common practice to assume that the factor, household and production accounts are endogenously determined and the government, capital and rest of the world accounts are exogenously determined. Figure 4.1 illustrates the partitioning of the SAM used for the model reported in this note, and identifies the notation.[2] The entries for the exogenous accounts are assumed to be aggregated, by row or column as appropriate, to be expressed as vectors. $\mathbf{A}_n$ is the matrix of 'technical' (column) coefficients for the endogenous accounts, $\mathbf{N}$ is the matrix of endogenous transactions, $\mathbf{y}_n$ is the vector of endogenous account totals, $\mathbf{x}$ is a vector of total exogenous demands for each of the endogenous accounts, $\mathbf{l}$ is the vector total exogenous leakages for each of the endogenous accounts and y_x is the total transactions for the exogenous accounts.

The materials balance equation for a SAM-Leontief model is

$$\mathbf{y}_n = \left[\mathbf{I} - \mathbf{A}_n\right]^{-1} \mathbf{x} = \mathbf{M}_a \mathbf{x} . \tag{1}$$

Multiplier analysis seeks to quantify the changes in the endogenous accounts consequent upon changes in the exogenous accounts totals subject to fixed 'technical' relationships, i.e.,

[2] The notation follows Lewis and Thorbecke (1992) and is based on Pyatt and Round (1979). The distinction here is that 'production1' can be conceived of 'commodity 1' and activity 1', etc.

$$\Delta \mathbf{y}_n = \mathbf{M}_a \Delta \mathbf{x} \,. \tag{2}$$

Embedded in this model is the assumption that all production sectors can increase output. This assumption is especially restrictive for agriculture, and a number of studies have questioned its appropriateness, e.g., Subramanian and Sadoulet (1990), Haggbalde *et al*., (1991) and Lewis and Thorbecke (1992). If the SAM is rearranged so that the production sectors that are assumed to be capacity constrained are ordered last within the endogenous accounts, the SAM can be represented as in Figure 4.2.

Figure 4.1

	Endogenous Accounts					Exogenous Accounts	Totals
	Factors	Households	Production 1	Production 2	Production 3		
Factors Households Production 1 Production 2 Production 3	$\mathbf{A}_n = \mathbf{N}.\hat{\mathbf{y}}_n^{-1}$					$\mathbf{x}$	$\mathbf{y}_n$
Exogenous Accounts	$\mathbf{a}_l = \mathbf{l}.\hat{\mathbf{y}}_n^{-1}$					t	y_x
Totals	$\mathbf{y}_n{}'$					y_x	

Adapted from Pyatt and Round (1979)

Figure 4.2

	Endogenous Accounts					Exogenous Accounts	Totals
	Factors	Households	Production 1	Production 2	Production 3		
Factors Households Production 1 Production 2	$\mathbf{A}_{nc} = \mathbf{N}_{nc}.\hat{\mathbf{y}}_{nc}^{-1}$				$\mathbf{Q} = \mathbf{N}_Q.\hat{\mathbf{y}}_c^{-1}$	$\mathbf{x}_{nc}$	$\mathbf{y}_{nc}$
Production 3	$\mathbf{R} = \mathbf{N}_R.\hat{\mathbf{y}}_{nc}^{-1}$				$\mathbf{A}_c = \mathbf{N}_c.\hat{\mathbf{y}}_c^{-1}$	$\mathbf{x}_c$	$\mathbf{y}_c$
Exogenous Accounts	$\mathbf{a}'_{l,nc} = \mathbf{l}'_{nc}.\hat{\mathbf{y}}_{nc}^{-1}$				$\mathbf{a}_{l,c} = \mathbf{l}_c.\hat{\mathbf{y}}_c^{-1}$	t	y_x
Totals	$\mathbf{y}_{nc}{}'$				$\mathbf{y}_c{}'$	y_x	

The subscript *nc* identifies the unconstrained endogenous accounts and *c* identifies the constrained accounts, in this case Production 3 account. Miller and Blair (1985, pp 325-333) develop such a mixed multiplier model in an input-output context, which Subramanian and

Sadoulet (1990) and Lewis and Thorbecke (1992) extend to a SAM-Leontief model.[3] If the SAM is rearranged so that the production sectors that are assumed to be capacity constrained are ordered last within the endogenous accounts, the materials balance equation, consistent with the SAM in Figure 4.2, can be written as

$$\begin{bmatrix} \mathbf{y}_{nc} \\ \mathbf{y}_{c} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{nc} & \mathbf{Q} \\ \mathbf{R} & \mathbf{A}_{c} \end{bmatrix} \begin{bmatrix} \mathbf{y}_{nc} \\ \mathbf{y}_{c} \end{bmatrix} + \begin{bmatrix} \mathbf{x}_{nc} \\ \mathbf{x}_{c} \end{bmatrix}. \tag{3}$$

The supply constraint means that $\mathbf{y}_c$ is exogenously fixed and hence $\mathbf{x}_c$ must be endogenously determined, whereas $\mathbf{x}_{nc}$ is exogenously fixed and $\mathbf{y}_{nc}$ is endogenously determined. Solving for the endogenously determined accounts gives, in multiplier form

$$\Delta \begin{bmatrix} \mathbf{y}_{nc} \\ \mathbf{x}_{c} \end{bmatrix} = \begin{bmatrix} (\mathbf{I} - \mathbf{A}_{n}) & \mathbf{0} \\ -\mathbf{R} & -\mathbf{I} \end{bmatrix}^{-1} . \begin{bmatrix} \mathbf{I} & \mathbf{Q} \\ \mathbf{0} & -(\mathbf{I} - \mathbf{A}_{c}) \end{bmatrix} . \Delta \begin{bmatrix} \mathbf{x}_{nc} \\ \mathbf{y}_{c} \end{bmatrix} = \mathbf{M}_{m} . \Delta \begin{bmatrix} \mathbf{x}_{nc} \\ \mathbf{y}_{c} \end{bmatrix}. \tag{4}$$

where $\Delta \mathbf{x}_{nc}$ is a consequence of changes in exogenous final demands, while $\Delta \mathbf{y}_{c}$ is a consequence of changes in the capacity of the constrained sectors.

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[3] An application of this model can be found in Berning and McDonald (2000).